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LIOUVILLE'S THEOREM AND THE RESTRICTED MEAN VALUE PROPERTY IN THE PLANE

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ABSTRACT. – Let U be a domain in $\mathbb{R}^2$ such that U^c is polar and let r be a real function on U such that $0 < r \le |\cdot| + M$. A positive numerical function f on U is called r-supermedian if, for every $x \in U$, the average of f on the disk of center x and radius $r(x)$ is at most $f(x)$. The purpose of this note is to give a short proof for the fact that every l.s.c. r-median function is constant. © Elsevier, Paris

1. Introduction

In the following let U be a domain in $\mathbb{R}^2$ such that U^c is polar (possibly empty), and let r be a real function on U such that

$$0 < r \le \rho := |\cdot| + M,$$

with $M \in \mathbb{R}_+$. Let λ be Lebesgue measure on $\mathbb{R}^2$ and, for all $x \in \mathbb{R}^2$ and $R > 0$, let $\lambda_{x,R}$ denote normed Lebesgue measure on the open disk $B(x, R) = \{z \in \mathbb{R}^2 : |z - x| < R\}$. A positive numerical function f on U is called *r–supermedian* (*r–median* resp.), if

$$\lambda^*_{x,r(x)}(f) \le f(x) \qquad (\lambda^*_{x,r(x)}(f) = f(x) \text{ resp.}),$$

for every $x \in U$. In [4] the following result is proved (under the more restrictive assumption that $B(x, r(x)) \subset U$ for every $x \in U$ if $U^c \neq \emptyset$):

THEOREM 1.1. – *Every l.s.c. r–supermedian function $f \ge 0$ on U is constant.*

And in [5] two corollaries are derived (as well as the corresponding statements on the real line):

COROLLARY 1.2. – *Let $f \ge 0$ be an r–supermedian function on U and assume that r is locally bounded away from zero. Then $f = \inf f(U)$ λ–a.e.*

COROLLARY 1.3. – *Let $f \ge 0$ be an r–median function on U. Assume that f is l.s.c. or that r is locally bounded away from zero. Then f is constant.*

The proof of Theorem 1.1 as given in [4] is very delicate. The statement is reduced to the case of domains having non-polar complement by removing a suitable small disk $\overline{V}$ where the value of f is almost maximal (assuming without loss of generality that f is bounded). This requires to replace measures $\lambda_{x,r(x)}$ for $x \in U \setminus \overline{V}$ with $B(x, r(x)) \cap V \neq \emptyset$

by more general means μ_x and to apply the general results of [3]. The basic idea is fairly simple, but the details are very technical.

The purpose of this note is to give a proof of Theorem 1.1 which is even simpler than the corresponding proof for continuous bounded r-median functions on $\mathbb{R}^d$, $d \geq 3$. Instead of considering lack of transience in the plane mainly as a nuisance causing additional problems, we shall exploit recurrence in U as an important element in our proof.

2. Proof of Theorem 1.1

Fix a l.s.c. r-supermedian function $f \geq 0$ on U, $f \not\equiv 0$, and real numbers $0 < a < \sup f(U)$, $0 < \varepsilon < 1$. In order to prove that f is constant it suffices to show that $f \geq (1-\varepsilon)a$.

We choose $x_0 \in U$ such that $f(x_0) > a$. There exists $0 < \alpha < \rho(x_0)$ such that the disk $A := B(x_0, \alpha)$ is contained in U and

$$f > a \qquad \text{on } A.$$

First we claim that there exists $\beta > 0$ such that, for every $x \in U$,

$$A \cap B(x, r(x)/2) = \emptyset \quad \text{or} \quad \lambda_{x,r(x)}(A) \geq \beta. \tag{2.1}$$

Indeed, consider $x \in U$ such that $A \cap B(x, r(x)/2) \neq \emptyset$. Then $A \cap B(x, r(x))$ contains a disk of diameter $\min(r(x)/2, \alpha)$, hence

$$\lambda_{x,r(x)}(A) \geq \left(\frac{1}{2}\min(\frac{1}{2}, \frac{\alpha}{r(x)})\right)^2.$$

Moreover, $|x - x_0| < \frac{r(x)}{2} + \alpha$, hence $r(x) \leq \rho(x) \leq \rho(x_0) + |x - x_0| < \rho(x_0) + \frac{r(x)}{2} + \alpha$,

$$r(x) < 2(\rho(x_0) + \alpha).$$

Thus (2.1) is satisfied with

$$\beta = \frac{1}{16}\left(\frac{\alpha}{\rho(x_0) + \alpha}\right)^2.$$

By proposition 2.4 in [2], (2.1) implies the existence of a constant $\delta > 0$ such that the following holds: If $x \in U$ and $0 \leq u \leq 1$ is a continuous solution of

$$\Delta u - Vu = 0, \qquad 0 \leq V \leq \delta\, 1_A \operatorname{dist}(\,\cdot\,, \partial B(x, r(x)))^{-2}$$

on $B(x, r(x))$ (where V is assumed to be Borel measurable), then

$$\int_{A^c} u \, d\lambda_{x,r(x)} \leq u(x). \tag{2.2}$$

Next we consider an exhaustion of the plane by disks

$$D_n := B(x_0, \rho(x_0)e^n) \qquad (n = 0, 1, 2, \ldots).$$

Of course, A is contained in D_0 and, for all $n \in \mathbb{N}$ and $x \in U \cap D_n$, we know that $r(x) \leq \rho(x) \leq \rho(x_0) + |x - x_0| \leq \rho(x_0) + \rho(x_0)e^n$. Since $2e^n + 1 = e^n(2 + e^{-n}) \leq e^{n+1}$ this shows that

$$B(x, r(x)) \subset D_{n+1}, \qquad \text{for all } n \in \mathbb{N} \text{ and } x \in D_n. \tag{2.3}$$

For every $n \in \mathbb{N}$, let u_n denote the solution of the Dirichlet problem

$$\Delta u - \delta 1_A \rho^{-2} u = 0 \quad \text{on } D_{n+1}, \qquad u = 1 \quad \text{on } \partial D_{n+1}.$$

The sequence (u_n) is decreasing to a continuous function u such that $0 \leq u \leq 1$ and

$$\Delta u - \delta 1_A \rho^{-2} u = 0 \qquad \text{on } \mathbb{R}^2.$$

In particular, u is a bounded subharmonic function on $\mathbb{R}^2$, hence constant. So $\delta 1_A \rho^{-2} u = \Delta u = 0$, $u = 0$ (see also [1]).

Fixing $y \in U$ we may choose $n \in \mathbb{N}$ such that $y \in D_n$ and

$$u_n(y) < \varepsilon/2.$$

We define a harmonic function h on $\mathbb{R}^2 \setminus \{x_0\}$ by

$$h(x) := \frac{1}{2} \ln \frac{|x - x_0|}{\rho(x_0)e^{n-1}} \qquad (x \neq x_0).$$

Obviously, $h = 0$ on ∂D_{n-1}, $h = 1$ on ∂D_{n+1}, and $h \geq 1/2$ on $D_{n+1} \setminus D_n$. Since u_n is harmonic on $D_{n+1} \setminus D_0$, the minimum principle yields that $u_n \geq h$ on $D_{n+1} \setminus \overline{D}_{n-1}$, hence

$$u_n \geq 1/2 \qquad \text{on } D_{n+1} \setminus D_n. \tag{2.4}$$

Moreover, for every $x \in D_n$ and $z \in B(x, r(x))$,

$$\text{dist}(z, \partial B(x, r(x))) = r(x) - |z - x| \leq \rho(x) - |z - x| \leq \rho(z),$$

hence, by (2.3) and (2.2),

$$\int_{A^c} u_n \, d\lambda_{x,r(x)} \leq u_n(x). \tag{2.5}$$

To conclude that $f(y) \geq (1 - \varepsilon)a$ it now suffices to apply the results on (transfinite) sweeping of measures obtained in [2]. Let Y be the one point compactification of D_{n+1}. By lemma 3.8 in [2], we may assume that r is Borel measurable so that, defining

$$P(x, \cdot) := \begin{cases} \lambda_{x,r(x)}, & \text{if } x \in (D_n \cap U) \setminus A, \\ \varepsilon_x, & \text{if } x \in A \cup (Y \setminus D_n) \cup (D_n \setminus U), \end{cases}$$

we obtain a Markov kernel on Y. Let $\mathcal{M}_{\mathcal{P}}(y)$ denote the smallest w*-closed set $\mathcal{N}$ of positive Radon measures on Y such that $\varepsilon_y \in \mathcal{N}$ and $P\nu := \int P(x, \cdot)\, \nu(dx) \in \mathcal{N}$ for every $\nu \in \mathcal{N}$. Since P is a Markov kernel, every $\nu \in \mathcal{M}_{\mathcal{P}}(y)$ is a probability measure. By proposition 3.2 and lemma 3.4 in [2], there exists a (unique) measure $\sigma_y \in \mathcal{M}_{\mathcal{P}}(y)$ such that

$$\sigma_y((D_n \cap U) \setminus A) = 0. \tag{2.6}$$

Defining

$$P'(x, \cdot) := \begin{cases} 1_{A^c} \lambda_{x,r(x)}, & \text{if } x \in (D_n \cap U) \setminus A, \\ \varepsilon_x, & \text{if } x \in A \cup (Y \setminus D_n) \cup (D_n \setminus U), \end{cases}$$

we similarly have $\mathcal{M}_{\mathcal{P}'}(y)$ and a measure $\sigma'_y \in \mathcal{M}_{\mathcal{P}'}(y)$ such that $\sigma'_y((D_n \cap U) \setminus A) = 0$. By lemma 3.6 in [2],

$$\sigma'_y(Y \setminus D_n) = \sigma_y(Y \setminus D_n). \tag{2.7}$$

Taking $u_n := 1$ on $Y \setminus D_{n+1}$ we know by (2.5) that

$$P' u_n \le u_n.$$

So, by proposition 3.3 in [2],

$$\sigma'_y(u_n) \le u_n(y) \le \varepsilon/2.$$

Since $u_n \ge 1/2$ on $Y \setminus D_n$, we conclude that $\sigma'_y(Y \setminus D_n) < \varepsilon$ and hence

$$\sigma_y(Y \setminus D_n) < \varepsilon \tag{2.8}$$

by (2.7). Moreover, there exists a positive superharmonic function s on D_{n+1} such that $s = \infty$ on the polar set $D_{n+1} \setminus U$ and $s(y) < \infty$. Taking $s := 0$ on $Y \setminus D_{n+1}$ we obviously have $Ps \le s$, hence $\sigma_y(s) \le s(y) < \infty$ showing that

$$\sigma_y(D_{n+1} \setminus U) = 0. \tag{2.9}$$

Combining (2.6), (2.8), (2.9) and using the fact that σ_y is a probability measure on Y we obtain that

$$\sigma_y(A) > 1 - \varepsilon. \tag{2.10}$$

Finally, we extend $f|_{D_{n+1}}$ by 0 to a l.s.c. function $v \ge 0$ on Y. Then $Pv \le v$, since f is r-supermedian. Therefore, applying proposition 3.3 of [2] a last time,

$$\sigma_y(v) \le v(y) = f(y). \tag{2.11}$$

Since $v = f > a$ on A, we conclude from (2.10) and (2.11) that

$$f(y) \ge \sigma_y(A)\, a \ge (1 - \varepsilon) a$$

finishing the proof.

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